

# GENERALIZED DOLBEAULT SEQUENCES IN PARABOLIC GEOMETRY

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ABSTRACT. In this paper, we show the existence of a sequence of invariant differential operators on a particular homogeneous model  $G/P$  of a Cartan geometry. The first operator in this sequence can be locally identified with the Dirac operator in  $k$  Clifford variables,  $D = (D_1, \dots, D_k)$ , where  $D_i = \sum_j e_j \cdot \partial_{ij} : C^\infty((\mathbb{R}^n)^k, \mathbb{S}) \rightarrow C^\infty((\mathbb{R}^n)^k, \mathbb{S})$ . We describe the structure of these sequences in case the dimension  $n$  is odd. It follows from the construction that all these operators are invariant with respect to the action of the group  $G$ .

These results are obtained by constructing homomorphisms of generalized Verma modules, what are purely algebraic objects.

## 1. MOTIVATION

There are two basic generalizations of the space of holomorphic functions to higher dimensions. One of them is the notion of holomorphic functions in several variables,  $f : \mathbb{R}^{2k} \simeq \mathbb{C}^k \rightarrow \mathbb{C}$ ,  $\bar{\partial}_j f = 0$  for  $j = 1, \dots, k$ . The second possible generalization deals with s.c. *monogenic functions*, which are defined on  $\mathbb{R}^n$  with values in the *Clifford algebra* or the *space of spinors* and solve the *Dirac equation*  $\sum_j e_j \cdot \partial_j f = 0$ . They have similar nice properties as holomorphic functions and coincide with them for  $n = 2$  ([9]).

Recently, many variations and generalizations of the classical Dirac operator appeared. While mathematical physicists study its spectra on different Riemannian spin-manifolds and other construct its analogs in non-riemannian geometries (see e.g. [17]), we may define the *Dirac operator in several Clifford variables* by  $D : C^\infty((\mathbb{R}^n)^k, \mathbb{S}) \rightarrow C^\infty((\mathbb{R}^n)^k, \mathbb{C}^k \otimes \mathbb{S})$ ,  $D = (D_1, \dots, D_k)$  (after identifying elements of the image with  $k$  spinor valued functions),  $D_i = \sum_j e_j \cdot \partial_{ij}$  where  $\mathbb{S}$  is the (usually complex) spinor space,  $x_{uv}$  the standard coordinates on  $(\mathbb{R}^n)^k$ ,  $u = 1, \dots, k$ ,  $j = 1, \dots, n$ , and  $\cdot$  the Clifford multiplication  $\mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{S}$ .

This is a common generalization of the space of holomorphic functions in several complex variables ( $n = 2$ ,  $k$  arbitrary) and the classical Dirac operator ( $k = 1$ ).

Many problems can be studied using a resolution of  $D$ , i.e. the (locally) exact complex of PDE's starting with the operator  $D$ . In the case of holomorphic functions in several complex variables,  $D$  being the Cauchy–Riemann operator ( $n = 2$ ,  $k$  arbitrary), this is just the Dolbeault sequence. For  $k = 2$ ,  $n$  even, the problem was studied in [11, 16]. However, for arbitrary  $n, k$ , the form of this resolution is not known yet, except of some special cases (see [4, 6, 7, 20]).

In this paper, the problem is treated in the framework of *parabolic geometry* and some particular results are obtained for  $n$  odd,  $k$  arbitrary. We construct sequences of differential operators starting with the Dirac operator  $D$  that are good candidates for being a resolution (the proof that they indeed form a resolution is still in progress). Our sequences contain all operators that are *invariant* with respect to the action of a quite large group and continue the Dirac operator.

Because the space of spinors arises naturally as a fundamental representation of the Lie group  $\text{Spin}(n)$ , it is natural to consider the Dirac operator as acting not only on  $C^\infty(\mathbb{R}^n, \mathbb{S})$  but rather on more general sections of a spinor bundle over a spin manifold  $M$  (see [8]). The simplest spin structure on the sphere  $S^n$  is the bundle  $\text{Spin}(n+1) \rightarrow \text{Spin}(n+1)/\text{Spin}(n) \simeq S^n$  and the associated spinor bundle is  $\text{Spin}(n+1) \times_{\text{Spin}(n)} \mathbb{S}$ . The usual Dirac operator acts between sections of this bundle and is invariant with respect to the group  $\text{Spin}(n+1)$  (the sections  $\Gamma(G \times_H \mathbb{V})$  can be naturally identified with invariant functions  $C^\infty(G, \mathbb{V})^H$  and the action of  $G$  is  $g \cdot f(x) := f(g^{-1}x)$ ). However, Dirac operator has a larger group of invariance. Whereas  $\text{Spin}(n+1)$  acts on the sphere by rotations, it is well known that Dirac operator is invariant with respect to all Möbius transformations. This is reflected by the fact that the bundle  $\text{Spin}(n+1) \rightarrow \text{Spin}(n+1)/\text{Spin}(n)$  is a reduction of a larger bundle  $\text{Spin}(n+1, 1)/P$ , where  $\text{Spin}(n+1, 1)$  acts on the null-cone of a form  $g$  of signature  $(n+1, 1)$  that defines the group  $\text{Spin}(n+1, 1)$ . The projectivisation of this null-cone is homeomorphic to the sphere  $S^n$  and  $P$  is the stabilizer of one line. It was shown in [10] that considering  $\mathbb{S}_1$  as a representation of  $P$  with highest weight

$$\begin{array}{c} \frac{n}{2} - 1 \quad 0 \quad \dots \quad 0 \quad \begin{array}{l} \nearrow 1 \\ \searrow 0 \end{array} \end{array} \quad \text{resp.} \quad \begin{array}{c} \frac{n}{2} - 1 \quad 0 \quad \dots \quad 0 \quad \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \end{array}$$

and  $\mathbb{S}_2$  a representation of  $P$  with highest weight

$$\begin{array}{c} \frac{n}{2} \quad 0 \quad \dots \quad 0 \quad \begin{array}{l} \nearrow 0 \\ \searrow 1 \end{array} \end{array} \quad \text{resp.} \quad \begin{array}{c} \frac{n}{2} \quad 0 \quad \dots \quad 0 \quad \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \end{array},$$

the Dirac operator is a  $\text{Spin}(n+1, 1)$ -invariant differential operator  $D : \Gamma(\text{Spin}(n+1, 1) \times_P \mathbb{S}_1) \rightarrow \Gamma(\text{Spin}(n+1, 1) \times_P \mathbb{S}_2)$ . In this sense, the Dirac operator is conformally invariant, as  $\text{Spin}(n+1, 1)$  (or, more exactly, its connected component) is the double-cover of the group of all Möbius transformations.

The subalgebra  $P$  is a *parabolic subalgebra* of  $G = \text{Spin}(n+1, 1)$ , i.e. its Lie algebra  $\mathfrak{p}$  contains a Borel algebra  $\mathfrak{b}$  of  $\mathfrak{g}$ , the Lie algebra of  $G$ . The bundle

$G \rightarrow G/P$  together with the Maurer-Cartan form on  $T(G)$  is an example of s.c. *parabolic geometry* (see [5, 18]).

In [10], an analogous construction is described for the group  $G = \text{Spin}(n + k, k)$  and  $P$  being a parabolic subgroup fixing a maximal vector subspace of the null cone of the metric of signature  $(n + k, k)$  defining  $\text{Spin}(n + k, k)$ . The reductive part of  $P$  is isomorphic to  $\text{GL}(k) \times \text{Spin}(n)$ . The Lie algebra  $\mathfrak{p}$  of  $P$  determines a gradation of the Lie algebra  $\mathfrak{g}$  of  $G$  so that  $\mathfrak{g} = \bigoplus_{j=-2}^2 \mathfrak{g}_j$  and  $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ . Again, choosing proper irreducible  $P$ -modules  $\mathbb{V}_1$  resp.  $\mathbb{V}_2$  with highest weights

$$\begin{array}{ccccccccccc} 0 & & & 0 & \frac{n}{2}-1 & 0 & & & 0 & 1 & \\ \circ & \cdots & \circ & \times & \circ & \cdots & \circ & \cdots & \circ & \rightarrow & \circ \\ 1 & 0 & & 0 & \frac{n}{2}-1 & 0 & & & 0 & 1 & \\ \circ & \cdots & \circ & \times & \circ & \cdots & \circ & \cdots & \circ & \rightarrow & \circ \end{array} \quad \text{resp.}$$

(and similar for  $n$  even), we showed in [10, 12] that there exists a  $G$ -invariant differential operator  $D : \Gamma(G \times_P \mathbb{V}_1) \rightarrow \Gamma(G \times_P \mathbb{V}_2)$  and, identifying local sections in the neighborhood of  $eP$  with  $\mathbb{V}_i$ -valued functions on the vector space  $\mathfrak{g}_- = \bigoplus_{j < 0} \mathfrak{g}_j$  in a natural way and restricting to functions that are constant in  $\mathfrak{g}_{-2} \subset \mathfrak{g}_-$ , this operator coincides with the Dirac operator in  $k$  Clifford variables (identifying  $\mathfrak{g}_{-1} \simeq (\mathbb{R}^n)^k$  as the adjoint representation of  $\mathfrak{g}_0 \simeq \mathfrak{gl}(k) \times \mathfrak{so}(n)$ ).

The question is, whether we can find sequences of  $G$ -invariant differential operators continuing the operator  $D$ . In case of the Dirac operator in one variable ( $k = 1$ ), this is not possible. We showed in [11] that for  $k = 2$ , there exist two further  $G$ -invariant differential operators so that they form a complex together with the first one.

In general, for any semisimple Lie group  $G$ , a parabolic subgroup  $P$  and some  $P$ -modules  $\mathbb{V}_1, \mathbb{V}_2$ , the  $G$ -invariant differential operators between sections of vector bundles  $D : \Gamma(G \times_P \mathbb{V}_1) \rightarrow \Gamma(G \times_P \mathbb{V}_2)$  are in 1 – 1 correspondence with the  $\mathfrak{g}$ -homomorphisms of generalized Verma modules  $M_{\mathfrak{p}}(\mathbb{V}_2^*) \rightarrow M_{\mathfrak{p}}(\mathbb{V}_1^*)$  induced by dual representations  $\mathbb{V}_2^*$  and  $\mathbb{V}_1^*$  (see [5]). Therefore, the generalized Verma modules and their homomorphisms will be studied in the rest of this paper.

## 2. BASICS ON VERMA MODULES

**2.1. Bruhat ordering.** Let us assume that  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{g}$ , i.e. a subalgebra containing the Borel subalgebra  $\mathfrak{b}$ . This induces a gradation  $\bigoplus_{j=-k}^k \mathfrak{g}_j$  of  $\mathfrak{g}$  so that  $\mathfrak{p} = \sum_{j \geq 0} \mathfrak{g}_j$ . Let  $\mathfrak{h}$  be a fixed Cartan subalgebra of  $\mathfrak{g}$  and  $\mathfrak{p}, \Phi^+$  a set of positive roots of  $\mathfrak{g}$  (and also of  $\mathfrak{p}$ ) and  $\Delta$  the set of simple roots, compatible with  $\Phi^+$ . There is a 1 – 1 correspondence between subsets  $\Sigma$  of  $\Delta$  and parabolic subalgebras  $\mathfrak{p}_{\Sigma} \subset \mathfrak{g}$ , where  $\mathfrak{p}_{\Sigma}$  contains the Cartan subalgebra, all positive root spaces and all those negative root spaces  $\mathfrak{g}_{-\beta}$ , such that  $\beta$  can be expressed as a sum of simple roots from  $\Delta - \Sigma$ . These roots form the set of simple roots of the algebra  $\mathfrak{g}_0$  from the associated

grading  $\oplus_{j=-k}^k \mathfrak{g}_j$ . In the Dynkin diagram, we draw the simple roots in  $\Sigma$  as crossed ( $\times$ ).

For any pair  $(\mathfrak{g}, \mathfrak{p})$  there exists a unique element  $E \in \mathfrak{g}$  called *grading element* so that  $\text{ad}(E)(X) = jX$  for any  $X \in \mathfrak{g}_j$ ,  $j = -k, \dots, k$ .

For each  $\beta \in \Phi^+$ , the *root reflection*  $s_\beta$  is a reflection in  $\mathfrak{h}^*$  fixing the hyperplane orthogonal to  $\beta$  in the Killing metric. In coordinates,  $s_\beta(\gamma) = \gamma - \gamma(H_\beta)\beta$  where  $H_\beta$  is the  $\beta$ -coroot (see e.g. [14]). The choice of  $\Delta$  determines the length  $l(w)$  of any element  $w$  of the Weyl group  $W$  of  $\mathfrak{g}$ . It is the minimal number  $k$  such that  $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_k}}$ ,  $\alpha_{i_j} \in \Delta$ ,  $s_{\alpha_{i_j}}$  being simple root reflections. This defines the Bruhat ordering on  $W$  in the following way:  $w \leq w'$  if and only if there exist  $w = w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_l = w'$ , where  $w_i \rightarrow w_{i+1}$  means that  $w_{i+1} = s_{\beta_i} w_i$  for some  $\beta_i \in \Phi^+$  and the length  $l(w_{i+1}) = l(w_i) + 1$ .

**2.2. Generalized Verma modules (GVM).** Let  $\mathbb{V}$  be a (usually finite dimensional) irreducible  $\mathfrak{p}$ -module with highest weight  $\lambda$ . The generalized Verma module (further GVM), introduced by Lepowski ([15]) is defined by  $M_{\mathfrak{p}}(\mathbb{V}) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}$ , where  $\mathcal{U}(\mathfrak{g})$  is the universal enveloping algebra of  $\mathfrak{g}$ , considered as a left  $\mathcal{U}(\mathfrak{g})$  and a right  $\mathcal{U}(\mathfrak{p})$ -module.  $M_{\mathfrak{p}}(\mathbb{V})$  is a highest weight module with highest weight  $\lambda$  and highest weight vector  $1 \otimes v_\lambda$ , where  $v_\lambda$  is a highest weight vector in  $\mathbb{V}$ . As a  $\mathfrak{g}_-$ -module and  $\mathfrak{g}_0$ -module,  $M_{\mathfrak{p}}(\mathbb{V}) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{V}$ . The GVM is uniquely determined by its highest weight  $\lambda$ , therefore we will sometimes denote the GVM with highest weight  $\lambda$  by  $M_{\mathfrak{p}}(\lambda + \delta)$ , where  $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$ . Assuming that  $\mathbb{V}$  is finite dimensional, the set of GVM's is isomorphic to the set of  $\mathfrak{p}$ -dominant and  $\mathfrak{p}$ -integral weights (this means weights  $\lambda$  such that  $\lambda(H_\alpha)$  is non-negative and integral for each  $\alpha \in \Delta - \Sigma$ ). Such weights will be further denoted by  $P_{\mathfrak{p}}^{++}$ .

If  $\mathfrak{p} = \mathfrak{b} = \mathfrak{h} \oplus_{\beta \in \Phi^+} \mathfrak{g}_\beta$  is the Borel subalgebra of  $\mathfrak{g}$ , the GVM  $M_{\mathfrak{b}}(\mathbb{V})$  is called true Verma module, or simply Verma module (in this case,  $\mathbb{V}$  is a one-dimensional representation of  $\mathfrak{b}$  and its weight can be any  $\lambda \in \mathfrak{h}^*$ ). Each highest weight module with highest weight  $\lambda$  is isomorphic to some factor of the Verma module  $M_{\mathfrak{b}}(\lambda + \delta)$ .

**2.3. Duality between GVM homomorphisms and invariant differential operators.** A  $G$ -invariant differential operator  $D : \Gamma(G \times_P \mathbb{V}) \rightarrow \Gamma(G \times_P \mathbb{W})$  is completely determined by the values  $Ds(eP)$  on sections ( $e \in G$  is the identity element). If the operator is of order  $k$ , the value  $Ds(eP)$  depends only on the  $k$ -jet  $J_{eP}^k s$  of a section  $s$  in  $eP$ . So, the operator  $D$  is determined by a map  $\tilde{D} : J_{eP}^k(G \times_P \mathbb{V}) \rightarrow \mathbb{W}$  that evaluates the image of a section  $s$  in  $eP$ , identifying the fiber over  $eP$  with  $\mathbb{W}$  in a natural way. More precisely,  $D(s)(eP) = [e, \tilde{D}(j_{eP}^k s)]_P$ . Because  $D$  is  $G$ -invariant,  $\tilde{D}$  has to be  $P$ -invariant, the action of  $P$  on the jets being the action on representatives.

The  $P$ -module  $J_{eP}^k(G \times_P \mathbb{V})$  of  $k$ -jets of sections is naturally isomorphic to the space of  $k$ -jets of  $P$ -invariant functions  $J_e^k(C^\infty(G, \mathbb{V})^P)$  (the action of  $P$  here being  $(p \cdot f)(x) = f(p^{-1}x)$ ). It can be shown that this is dual, as a  $P$ -module, to  $\mathcal{U}_k(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}^*$  (where  $\mathcal{U}_k(\mathfrak{g})$  is the  $k$ -th filtration of  $\mathcal{U}(\mathfrak{g})$ ) and the duality is given by

$$(2.1) \quad (Y_1 \dots Y_l \otimes_{\mathcal{U}(\mathfrak{p})} A)(j_e^k f) := A((L_{Y_1} \dots L_{Y_l} f)(e))$$

for  $l \leq k$ ,  $A \in \mathbb{V}^*$ ,  $j_e^k f$  the  $k$ -jet of  $f$  in  $e$ ,  $Y_j \in \mathfrak{g}$  and  $L_{Y_j}$  the derivation with respect to the left invariant vector fields on  $G$  induced by  $Y_j$  (see [5] for details).

Any  $P$ -homomorphism  $\tilde{D} : J_e^k(C^\infty(G, \mathbb{V})^P) \rightarrow \mathbb{W}$  is determined by its dual map  $\tilde{D}^* : \mathbb{W}^* \rightarrow J_e^k(C^\infty(G, \mathbb{V})^P)^*$  and we see from (2.1) that the right hand side can be identified with a  $P$ -submodule of  $M_{\mathfrak{p}}(\mathbb{V}^*)$ . Further, each  $P$ -homomorphism  $\tilde{D}^* : \mathbb{W}^* \rightarrow M_{\mathfrak{p}}(\mathbb{V}^*)$  can be extended to a  $(\mathfrak{g}, P)$ -homomorphism  $M_{\mathfrak{p}}(\mathbb{W}^*) \rightarrow M_{\mathfrak{p}}(\mathbb{V}^*)$  of GVM's by  $y_1 \dots y_l \otimes v \mapsto y_1 \dots y_l \tilde{D}^*(v)$  for  $y_j \in \mathfrak{g}_-$ , the action of  $\mathfrak{p}$  on  $\mathbb{W}^*$  being the infinitesimal action of  $P$  (we identified  $M_{\mathfrak{p}}(\mathbb{W}^*) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{W}^*$ ).

It follows that there is a duality between invariant linear differential operators  $D : \Gamma(G \times_P \mathbb{V}) \rightarrow \Gamma(G \times_P \mathbb{W})$  of any finite order and  $(\mathfrak{g}, P)$ -homomorphisms of GVM's  $M_{\mathfrak{p}}(\mathbb{W}^*) \rightarrow M_{\mathfrak{p}}(\mathbb{V}^*)$ . Note that, if the inducing representations  $\mathbb{V}$  and  $\mathbb{W}$  are both  $P$ -modules, then each  $\mathfrak{g}$ -homomorphism  $M_{\mathfrak{p}}(\mathbb{V}) \rightarrow M_{\mathfrak{p}}(\mathbb{W})$  is a  $(\mathfrak{g}, P)$ -homomorphism as well.

Finally, note that if the Lie groups  $(G, P)$  are real but the representation spaces  $\mathbb{V}, \mathbb{W}$  are complex representations of  $P$ , then the real GVM  $M_{\mathfrak{p}}(\mathbb{V})$  is  $(\mathfrak{g}-)$  isomorphic to the complex GVM induced by  $\mathbb{V}$ , considered as a complex representation of the complexified Lie algebra  $\mathfrak{p}^{\mathbb{C}}$ . Therefore, we may restrict to GVM's associated to complex Lie algebras  $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{p}^{\mathbb{C}})$ .

**2.4. Homomorphisms of GVM's.** The GVM's are highest weight modules, therefore they admit central characters. As each  $\mathfrak{g}$ -homomorphism of highest weight modules must preserve the central character, it follows from Harris-Chandra theorem (see, e.g. [14]) that a  $\mathfrak{g}$ -homomorphism  $M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  may exist only if  $\mu$  and  $\lambda$  are on the same orbit of the Weyl group  $W$  of the Lie algebra  $\mathfrak{g}$ . (Recall that the highest weights of these modules are  $\mu - \delta$  and  $\lambda - \delta$ .) For  $\lambda \in P_{\mathfrak{p}}^{++} + \delta$ , there exist only a finite number of weights  $\mu \in P_{\mathfrak{p}}^{++} + \delta$  on the same orbit of the Weyl group.

In the case of true Verma modules, there is a classification of their homomorphisms, done by Verma and Bernstein-Gelfand-Gelfand ([1, 2, 21]), summarized in the following statements:

**Theorem 2.4.1.** *Let  $\mu, \lambda \in \mathfrak{h}^*$ . Each homomorphism  $M_{\mathfrak{b}}(\mu) \rightarrow M_{\mathfrak{b}}(\lambda)$  is injective and  $\dim(\text{Hom}(M_{\mathfrak{b}}(\mu), M_{\mathfrak{b}}(\lambda))) \leq 1$ . Therefore, we can write  $M_{\mathfrak{b}}(\mu) \subset M_{\mathfrak{b}}(\lambda)$  in such case.*

A nonzero homomorphism of Verma modules  $M_{\mathfrak{b}}(\mu) \rightarrow M_{\mathfrak{b}}(\lambda)$  exists if and only if there exist weights  $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$  so that  $\lambda_{i+1} = s_{\beta_i} \lambda_i$  for some positive roots  $\beta_i$  and  $\lambda_i(H_{\beta_i}) \in \mathbb{N}$  for all  $i$  ( $s_{\beta} \in W$  is the  $\beta$ -root reflection). Equivalently,  $\lambda_i - \lambda_{i-1}$  is a positive integral multiple of some positive root for all  $i$ .

Let  $\lambda \in P_{\mathfrak{g}}^{++} + \delta$  (i.e.  $\lambda - \delta$  is  $\mathfrak{g}$ -dominant and  $\mathfrak{g}$ -integral). Then there exists a nonzero homomorphism  $M_{\mathfrak{b}}(w'\lambda) \rightarrow M_{\mathfrak{b}}(w\lambda)$  if and only if  $w \leq w'$  in the Bruhat ordering.

If  $\lambda$  is only  $\mathfrak{g}$ -dominant ( $\lambda(H_{\beta}) > 0$  for all  $\beta \in \Phi^+$ ), then the existence of a nonzero standard homomorphism  $M_{\mathfrak{b}}(w'\lambda) \rightarrow M_{\mathfrak{b}}(w\lambda)$  still implies  $w \leq w'$  in the Bruhat ordering (but not the opposite).

Because  $M_{\mathfrak{p}}(\lambda)$  is a highest weight module, it is isomorphic to a factor of true Verma module  $M_{\mathfrak{b}}(\lambda)/W$ . It was proved by Lepowski that  $W \simeq \sum_{\alpha \in \Delta - \Sigma} M_{\mathfrak{b}}(s_{\alpha}\lambda)$  ( $\Sigma \subset \Delta$  determines the parabolic subalgebra  $\mathfrak{p}$  and all the modules  $M_{\mathfrak{b}}(s_{\alpha}\lambda)$  are considered as submodules of  $M_{\mathfrak{b}}(\lambda)$ ). A homomorphism  $M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  is called standard, if it is a factor of a true Verma module homomorphism  $M_{\mathfrak{b}}(\mu) \rightarrow M_{\mathfrak{b}}(\lambda)$ . Up to multiple, there exists at most one standard homomorphism from  $M_{\mathfrak{p}}(\mu)$  to  $M_{\mathfrak{p}}(\lambda)$ . The following is known about standard homomorphisms of GVM's:

**Theorem 2.4.2.** *Let  $\mu, \lambda \in P_{\mathfrak{p}}^{++} + \delta$ ,  $i : M_{\mathfrak{b}}(\mu) \rightarrow M_{\mathfrak{b}}(\lambda)$  be a homomorphism of Verma modules. Then the standard homomorphism  $M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  is zero if and only if there exists  $\alpha \in \Delta - \Sigma$  so that  $i(M_{\mathfrak{b}}(\mu)) \subset M_{\mathfrak{b}}(s_{\alpha}\lambda)$  (identifying  $M_{\mathfrak{b}}(s_{\alpha}\lambda)$  with a submodule of  $M_{\mathfrak{b}}(\lambda)$ ).*

Let us denote by  $W_{\mathfrak{p}}$  the subgroup of  $W$  generated by simple root reflections  $\{s_{\alpha}, \alpha \in \Delta - \Sigma\}$  and  $W^{\mathfrak{p}}$  the subset of  $W$  consisting of those  $w \in W$  so that  $w\tilde{\lambda}$  is  $\mathfrak{p}$ -dominant for each  $\mathfrak{g}$ -dominant weight  $\tilde{\lambda}$ . Any  $w \in W$  can be uniquely decomposed  $w = w_p w^{\mathfrak{p}}$  where  $w_p \in W_{\mathfrak{p}}$  and  $w^{\mathfrak{p}} \in W^{\mathfrak{p}}$  and the length  $l(w) = l(w_p) + l(w^{\mathfrak{p}})$ . We define the *parabolic Hasse graph* for  $(\mathfrak{g}, \mathfrak{p})$  to be the set  $W^{\mathfrak{p}}$  of vertices with arrows  $w \rightarrow w'$  if and only if  $w \rightarrow w'$  in  $W$ .

The following two properties of the parabolic Hasse graph will be used later (for the proof, see [3]):

**Lemma 2.4.3.** (1) *If  $w' = s_{\gamma}w$ , then either  $w \leq w'$  or  $w' \leq w$  in the Bruhat ordering.*

(2) *Let  $w, w' \in W^{\mathfrak{p}}$  and  $w \leq w'$  in the Bruhat ordering. Then there exists a path  $w \rightarrow w_1 \rightarrow \dots \rightarrow w_n \rightarrow w'$  so that all  $w_i$  are in  $W^{\mathfrak{p}}$ .*

The following theorem can be used to prove the existence of a standard GVM homomorphism:

**Theorem 2.4.4.** *Let  $\tilde{\lambda}$  be a strictly dominant weight (i.e.  $\tilde{\lambda}(H_\beta) > 0$  for  $\beta \in \Phi^+$ ),  $w, w' \in W^p$ ,  $w \rightarrow w'$  in the parabolic Hasse graph for  $(\mathfrak{g}, \mathfrak{p})$  and assume that  $w\tilde{\lambda}, w'\tilde{\lambda} \in P_{\mathfrak{p}}^{++} + \delta$ . Further, suppose that there exists a nonzero homomorphism of true Verma modules  $M_{\mathfrak{b}}(w'\tilde{\lambda}) \rightarrow M_{\mathfrak{b}}(w\tilde{\lambda})$ . Then the standard homomorphism  $M_{\mathfrak{p}}(w'\tilde{\lambda}) \rightarrow M_{\mathfrak{p}}(w\tilde{\lambda})$  is nonzero.*

**Remark 2.4.5.** *In [15], the theorem is formulated only for  $\tilde{\lambda} \in P^{++} + \delta$ , but the proof works for non-integral  $\tilde{\lambda}$  as well. Note, that for non-integral (and neither  $\mathfrak{g}$ -, nor  $\mathfrak{p}$ -dominant)  $\tilde{\lambda} - \delta$ , the weights  $w\tilde{\lambda} - \delta$  and  $w'\tilde{\lambda} - \delta$  may still be  $\mathfrak{p}$ -dominant and  $\mathfrak{p}$ -integral.*

*Proof.* Assume that the standard homomorphism is zero. It follows from lemma 2.4.2 that there exists  $\alpha \in \Delta - \Sigma$  so that  $M_{\mathfrak{b}}(w'\tilde{\lambda}) \subset M_{\mathfrak{b}}(s_\alpha w\tilde{\lambda})$ . The last statement of theorem 2.4.1 implies that  $w' > s_\alpha w$  in the Bruhat ordering. But, because  $w\tilde{\lambda} \in P_{\mathfrak{p}}^{++} + \delta$  and  $\alpha \in \Delta - \Sigma$ , we have  $(w\tilde{\lambda})(H_\alpha) \in \mathbb{N}$  and it follows from 2.4.1 that  $M_{\mathfrak{b}}(s_\alpha w\tilde{\lambda}) \subset M_{\mathfrak{b}}(w\tilde{\lambda})$  and  $l(s_\alpha w) = l(w) + 1$ . So we have  $l(w') > l(s_\alpha w) > l(w)$  which contradicts  $l(w') = l(w) + 1$ .  $\square$

For any weight  $\lambda$ , there always exists a dominant weight  $\tilde{\lambda}$  (i.e.  $\tilde{\lambda}(H_\beta) \geq 0$  for  $\beta \in \Phi^+$ ) on the same orbit of the Weyl group. If there exists some  $\beta$  so that  $\tilde{\lambda}(H_\beta) = 0$ , we say that the generalized Verma modules  $M_{\mathfrak{p}}(w\tilde{\lambda})$  have *singular character* and the weights  $w\tilde{\lambda}$  are called *singular*. Theorem 2.4.4 cannot be generalized to singular weights, because for singular  $\tilde{\lambda}$ , the weight  $w\tilde{\lambda}$  doesn't determine  $w$  uniquely. (However, there are indications that a similar theorem may be true, if we admit non-standard GVM homomorphisms.)

The following lemma will be used for comparing lengths of two elements in  $W^p$ :

**Lemma 2.4.6.** *Let  $E$  be the grading element for the pair  $(\mathfrak{g}, \mathfrak{p})$  and let  $w, w' \in W^p$ ,  $w' = s_\gamma w$  and  $l(w') > l(w)$ . Then  $w\delta(E) - w'\delta(E) \in \mathbb{N}$ .*

*Proof.* Because  $w \in W^p$  and  $w' = s_\gamma w \in W^p$ , the uniqueness of the decomposition  $W = W_p W^p$  yields  $s_\gamma \notin W_p$ . From the definition,  $W_p = W_{\mathfrak{g}_0}$ , the Weyl group of  $\mathfrak{g}_0$ , so the root  $\gamma$  cannot be expressed as sum of simple roots in  $\Delta - \Sigma$ . The definition of the grading  $\oplus_j \mathfrak{g}_j$  of  $\mathfrak{g}$ , associated to the pair  $(\mathfrak{g}, \mathfrak{p})$  implies that the  $\gamma$ -root space generator  $X_\gamma \in \mathfrak{g}_i$  for some  $i > 0$ , so  $\gamma(E) = i \in \mathbb{N}$ . We obtain  $w'\delta(E) = (s_\gamma w\delta)(E) = (w\delta - w\delta(H_\gamma)\gamma)(E) = w\delta(E) - iw\delta(H_\gamma)$ . Because  $\delta$  is dominant and  $l(w') > l(w)$ , we have  $w\delta(H_\gamma) > 0$ . The weight  $\delta$  is also integral, because  $\delta(H_\alpha) = 1$  for each  $\alpha \in \Delta$ . So the difference  $(w\delta - w'\delta)(E) = iw\delta(H_\gamma)$  is a product of two positive integers.  $\square$

**2.5. Order of the differential operator dual to a GVM homomorphism.** The following theorem is an important tool for determining the

order of an operator, dual to a homomorphism of generalized Verma modules, if the highest weights of the inducing representations are known.

**Theorem 2.5.1.** *Let  $\mu, \lambda$  be highest weights of some irreducible finite-dimensional  $P$ -modules  $\mathbb{V}_\mu, \mathbb{V}_\lambda$  and  $\phi : M_{\mathfrak{p}}(\mathbb{V}_\mu) \rightarrow M_{\mathfrak{p}}(\mathbb{V}_\lambda)$  be a nonzero homomorphism of generalized Verma modules. Let  $E$  be the grading element for  $(\mathfrak{g}, \mathfrak{p})$  and let  $o := (\lambda - \mu)(E)$ . Then  $o$  is an integer larger or equal to the order of the dual differential operator  $\Gamma(G \times_P \mathbb{V}_\lambda^*) \rightarrow \Gamma(G \times_P \mathbb{V}_\mu^*)$ . Further, if  $o \in \{1, 2\}$ , then  $o$  is the order of the operator.*

*Proof.* Let  $v_\mu$  be the highest weight vector of  $\mathbb{V}_\mu$  and  $\phi(1 \otimes v_\mu) = \sum_j y_j \otimes v_j$ ,  $y_j \in \mathcal{U}(\mathfrak{g}_-)$ ,  $v_j \in \mathbb{V}_\lambda$  ( $M_{\mathfrak{p}}(\lambda) \simeq \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$  as vector space). Let  $k$  be the maximal integer so that  $y_i \in \mathcal{U}_k(\mathfrak{g}_-) - \mathcal{U}_{k-1}(\mathfrak{g}_-)$  for some  $y_i$  and let  $0 \neq g_0 \in \mathcal{U}(\mathfrak{g}_0)$ . Then  $\phi$  maps  $1 \otimes g_0 \cdot v_\mu = g_0 \otimes_{\mathcal{U}(\mathfrak{p})} v_\mu$  to

$$\sum_j g_0 y_j \otimes_{\mathcal{U}(\mathfrak{p})} v_j = \sum_j (y_j g_0 + [g_0, y_j]) \otimes_{\mathcal{U}(\mathfrak{p})} v_j = \sum_j (y_j \otimes g_0 \cdot v_j + [g_0, y_j] \otimes v_j)$$

because for  $g_0 \in \mathcal{U}(\mathfrak{g}_0)$  and  $y_j \in \mathcal{U}(\mathfrak{g}_-)$ ,  $[g_0, y_j] \in \mathcal{U}(\mathfrak{g}_-)$  ( $[a, b] = ab - ba$  is the commutator in the associative algebra  $\mathcal{U}(\mathfrak{g})$ ). Simple commutation relations show that, if  $y_j \in \mathcal{U}_l(\mathfrak{g}_-) - \mathcal{U}_{l-1}(\mathfrak{g}_-)$ , then  $[g_0, y_j] \in \mathcal{U}_l(\mathfrak{g}_-) - \mathcal{U}_{l-1}(\mathfrak{g}_-)$  as well. Therefore,  $\phi$  maps  $1 \otimes g_0 \cdot v_\mu$  into  $\mathcal{U}_k(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$  but not to  $\mathcal{U}_{k-1}(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$ .  $\mathbb{V}_\mu$  is an irreducible  $\mathfrak{p}$ -module and  $\mathfrak{g}_0$  is the reductive part of  $\mathfrak{p}$ , so  $\mathcal{U}(\mathfrak{g}_0)v_\mu = \mathbb{V}_\mu$  and  $\phi$  maps  $1 \otimes v_\mu$  into  $\mathcal{U}_k(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$ . Let  $v \in \mathbb{V}_\mu$ ,  $\phi(1 \otimes v) = \sum_j \tilde{y}_j \otimes \tilde{v}_j$ ,  $\tilde{v}_j \in \mathbb{V}_\lambda$ ,  $\tilde{y}_j \in \mathcal{U}_k(\mathfrak{g}_-)$  and  $\tilde{y}_i \notin \mathcal{U}_{k-1}(\mathfrak{g}_-)$  for some  $i$ . Let  $\tilde{y}_j = y_1^{(j)} \dots y_{l(j)}^{(j)}$  for some  $y_u^{(j)} \in \mathfrak{g}_-$ ,  $l(j) \leq k$  and  $l(i) = k$ .

Applying the duality (2.1), the differential operator  $D$  satisfies

$$v((Df)(0)) = \sum_j \tilde{v}_j (L_{y_1^{(j)}} \dots L_{y_{l(j)}^{(j)}} (f)(0)),$$

where  $L_{y_u^{(j)}}$  are the left invariant vector fields generated by  $y_u^{(j)} \in \mathfrak{g}_-$ . So, the operator  $D$  dual to the homomorphism is of order  $k$ .

Let us suppose that the operator has order  $k$ , i.e.  $\phi$  maps  $1 \otimes v_\mu$  into  $\mathcal{U}_k(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$  but not into  $\mathcal{U}_{k-1}(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$ . Let  $\{y_1, \dots, y_n\}$  be an ordered basis of  $\mathfrak{g}_-$  that consists of generators of negative root spaces in  $\mathfrak{g}_-$ .

Let  $\phi(1 \otimes v_\mu) = \sum_j \tilde{y}_j \otimes v_j$  and assume that all the  $v_j$ 's are weight vectors in  $\mathbb{V}_\lambda$  and  $\tilde{y}_j$  is a product of the  $y_j$ 's (it follows from the PBW theorem that such expression is always possible). Then all  $\tilde{y}_j \otimes v_j$  are weight vectors and, because their sum is a weight vector of weight  $\mu$ , each  $\tilde{y}_j \otimes v_j$  is a weight vector of weight  $\mu$  as well.

Because  $\phi(1 \otimes v_\mu) \notin \mathcal{U}_{k-1}(\mathfrak{g}_-) \otimes \mathbb{V}_\lambda$ , there exists  $i$  such that  $\tilde{y}_i = y_{i_1} \dots y_{i_k}$  is a product of  $k$  elements. Let  $u_j \in \mathbb{N}$  be defined by  $y_{i_j} \in \mathfrak{g}_{-u_j}$ . The action



of the grading element on  $y_{i_1} \dots y_{i_k} \otimes v_i$  is

$$\begin{aligned} E \cdot (y_{i_1} \dots y_{i_k} \otimes v_i) &= E y_{i_1} \dots y_{i_k} \otimes_{\mathcal{U}(\mathfrak{p})} v_i = \\ &= (y_{i_1} E + [E, y_{i_1}]) y_{i_2} \dots y_{i_k} \otimes_{\mathcal{U}(\mathfrak{p})} v_i = \dots = \\ &= y_{i_1} \dots y_{i_k} (\lambda(E) - u_1 - \dots - u_k) \otimes v_i \end{aligned}$$

But  $y_{i_1} \dots y_{i_k} \otimes v_i$  is a weight vector of weight  $\mu$ , so the left hand side equals  $\mu(E)(y_{i_1} \dots y_{i_k} \otimes v_i)$ . It follows

$$(2.2) \quad (\lambda - \mu)(E) = \sum_j u_j \geq k$$

because  $u_j \geq 1$  for all  $j$ . So, we see that  $(\lambda - \mu)(E)$  is always an integer larger or equal to the order of the operator.

It follows immediately that  $(\lambda - \mu)(E) = 1$  implies that the operator is of first order. To finish the proof, it remains to show that for a first order operator,  $(\lambda - \mu)(E)$  is 1.

Assume that  $D$  is an operator of first order. This means that  $\phi(1 \otimes v_\mu) = \sum_j y_j \otimes v_j$  for  $y_j \in \mathcal{U}_1(\mathfrak{g}_-)$  and again, assume that  $y_j$  are either constants or generators of negative root spaces and  $v_i$  are weight vectors. All the terms  $y_j \otimes v_j$  are of weight  $\mu$ , and therefore,

$$\mu(E)(y_j \otimes v_j) = E(y_j \otimes v_j) = (\lambda(E) + [E, y_j])(y_j \otimes v_j)$$

so  $[E, y_j] = (\mu - \lambda)(E)$  for all  $j$  and it follows that all the  $y_j$ 's are from the same graded components of  $\mathfrak{g}$ . If  $y_j \in \mathfrak{g}_{-1}$ , so  $(\lambda - \mu)(E) = 1$  and we are done. Assume, for contradiction, that  $y_j \in \mathfrak{g}_{-k}$  for  $k > 1$ .

Because  $\sum_j y_j \otimes v_j \in \mathfrak{g}_{-k} \otimes \mathbb{V}_\lambda$ , choosing a basis  $\{\tilde{v}_1, \dots, \tilde{v}_m\}$  of  $\mathbb{V}_\lambda$ ,  $\sum_j y_j \otimes v_j$  can be uniquely expressed as  $\sum_{j=1}^m \tilde{y}_j \otimes \tilde{v}_j$  for some  $\tilde{y}_j \in \mathfrak{g}_{-k}$ . Because it is a homomorphic image of a highest weight vector in  $M_{\mathfrak{p}}(\mu)$ , it must be annihilated by all positive root spaces in  $\mathfrak{g}$ , in particular, by any generator  $x$  of a root space in  $\mathfrak{g}_1$ :

$$\begin{aligned} x \cdot \left( \sum_j \tilde{y}_j \otimes \tilde{v}_j \right) &= \sum_j x \tilde{y}_j \otimes_{\mathcal{U}(\mathfrak{p})} \tilde{v}_j = \sum_j (\tilde{y}_j x + [x, \tilde{y}_j]) \otimes_{\mathcal{U}(\mathfrak{p})} \tilde{v}_j = \\ &= \sum_j (\tilde{y}_j \otimes_{\mathcal{U}(\mathfrak{p})} x \cdot \tilde{v}_j + [x, \tilde{y}_j] \otimes_{\mathcal{U}(\mathfrak{p})} \tilde{v}_j) = \sum_j [x, \tilde{y}_j] \otimes \tilde{v}_j = 0 \end{aligned}$$

because  $[x, \tilde{y}_j] \in \mathfrak{g}_{-k+1} \subset \mathfrak{g}_-$  and  $x \cdot \tilde{v}_\lambda = 0$ . Because  $\tilde{v}_j$  forms a basis of  $\mathbb{V}_\mu$ , it follows that for each  $j$ ,  $[x, \tilde{y}_j] = 0$  for all  $x \in \mathfrak{g}_1$ . The grading fulfills that  $\mathfrak{g}_{-1}$  generates  $\mathfrak{g}_-$  and  $\mathfrak{g}_1$  generates  $\mathfrak{p}^+ = \sum_{i \geq 1} \mathfrak{g}_i$ . The Jacobi identity implies that if  $\tilde{y}_j$  commutes with  $\mathfrak{g}_1$ , it commutes with all the  $\mathfrak{p}^+$  as well. Let  $\tilde{y}_j = \sum_i a_i y_{-\phi_i}$  where  $y_{-\phi_i}$  is a generator of the  $-\phi_i$ -root space. Define  $x := \sum_i a_i x_{\phi_i}$ , where  $x_{\phi_i}$  is a generator of the  $\phi_i$ -root space. We see that  $x \in \mathfrak{g}_k$  and  $[x, \tilde{y}_j] = \sum_i a_i^2 [x_\phi, y_{-\phi}] \neq 0$  and we have a contradiction.  $\square$

### 3. THE ORBITS ASSOCIATED WITH THE DIRAC OPERATOR

**3.1. Existence of the homomorphisms.** Let us suppose that  $n$  is odd,  $\mathfrak{g} = B_{k+(n-1)/2} = \mathfrak{so}(n+2k, \mathbb{C})$ ,  $\mathfrak{p}$  its parabolic subalgebra corresponding to

$$\circ \text{---} \dots \text{---} \circ \times \text{---} \circ \text{---} \dots \text{---} \circ \Rightarrow \circ$$

where the  $k$ -th node is crossed ( $\Sigma = \{\alpha_k\}$ ). The subalgebra  $\mathfrak{p}$  induces the

2-gradation  $\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$ , where  $\mathfrak{g}_0$  consists of blocks of dimension  $k \times k$ ,  $n \times n$  and  $k \times k$ . The corresponding grading element is  $E = \text{diag}(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$  and the action of a weight  $[a_1, \dots, a_k | b_1, \dots, b_{(n-1)/2}]$  on  $E$  is  $\sum_i a_i$ .

In this section, we will try to describe the structure of GVM homomorphisms on the Weyl orbit of the weight

$$\lambda = \overset{0}{\circ} \text{---} \dots \text{---} \overset{0}{\circ} \times \overset{-\frac{n}{2}}{\circ} \text{---} \dots \text{---} \overset{0}{\circ} \Rightarrow \overset{1}{\circ} + \delta.$$

It was shown in [10, 11] that there exists a GVM homomorphism  $M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  so that the dual differential operator may be identified with the Dirac operator in various Clifford variables, as noticed in the introduction (choosing the real Lie groups  $G = \text{Spin}(n+k, k)$  and  $P$  the parabolic subgroup so that its complexified Lie algebra is  $\mathfrak{p}$ ).

Let us represent the elements of  $\mathfrak{g}$  as matrices antisymmetric with respect to the anti-diagonal, choose the Cartan algebra to be the algebra of diagonal matrices in  $\mathfrak{g}$  and a natural basis  $\{\epsilon_i\}$  of  $\mathfrak{h}^*$  defined by

$$\epsilon_i(\text{diag}(a_1, \dots, a_{k+(n-1)/2}, 0, -a_{k+(n-1)/2}, \dots, -a_1)) := a_i$$

(see e.g. [13] for details).

In the  $\epsilon_i$ -basis,  $\delta = [\dots, 5/2, 3/2, 1/2]$ ,  $\mathfrak{g}$ -dominant weights are those  $[a_1, \dots, a_{k+(n-1)/2}]$  such that  $a_1 \geq a_2 \geq \dots \geq a_{k+(n-1)/2} \geq 0$  and  $\mathfrak{p}$ -dominant weights must fulfill  $a_1 \geq a_2 \geq \dots \geq a_k$  and  $a_{k+1} \geq \dots \geq a_{k+(n-1)/2} \geq 0$ . A weight is  $\mathfrak{p}$ -dominant and  $\mathfrak{p}$ -integral, if, moreover,  $a_i - a_j \in \mathbb{Z}$  for  $i, j \leq k$  and  $a_l \in \mathbb{Z}/2$  for  $l > k$ . Positive roots are all  $[0, \dots, 0, 1, 0, \dots, -1, \dots]$ ,  $[\dots, 1, \dots, 1, \dots]$  and  $[\dots, 0, 1, 0, \dots]$ . The corresponding root reflections map the weight  $[\dots, a_i, \dots, a_j, \dots]$  to  $[\dots, a_j, \dots, a_i, \dots]$  (transpositions), or to  $[\dots, -a_j, \dots, -a_i, \dots]$  (sign-transpositions) or to  $[\dots, -a_i, \dots, a_j, \dots]$  (sign-change).

The weight  $\lambda$  we consider can be written in the  $\epsilon_i$ -basis as

$$\lambda = [(2k-1)/2, \dots, 3/2, 1/2 | \dots, 3, 2, 1].$$

**Lemma 3.1.1.** *Let  $k = 2$ . Then there exist three nonzero weights  $\mu, \nu, \xi \in P_{\mathfrak{p}}^{++}$  on the orbit of  $\lambda$  and nonzero standard homomorphisms*

$$M_{\mathfrak{p}}(\xi) \rightarrow M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda),$$

where the weights are described by the following diagram:

- $[3/2, 1/2 | \dots, 2, 1] = \lambda$
- |
- $[3/2, -1/2 | \dots, 2, 1] = \mu$
- |
- $[1/2, -3/2 | \dots, 2, 1] = \nu$
- |
- $[-1/2, -3/2 | \dots, 2, 1] = \xi$

*Proof.* The existence of true Verma module homomorphisms  $M_{\mathfrak{b}}(\xi) \rightarrow \dots \rightarrow M_{\mathfrak{b}}(\lambda)$  follows easily from Theorem 2.4.1. All the weights are from  $P_{\mathfrak{p}}^{++} + \delta$  and they are on the orbit of the  $\mathfrak{g}$ -dominant weight  $\tilde{\lambda} = [\dots, 4, 3, 2, 3/2, 1, 1/2]$ . This weight is nonsingular, because its coefficients are strictly decreasing and the last one is strictly positive.

Let  $w$  resp.  $w', w'', w'''$  be the elements of  $W$  that takes  $\tilde{\lambda}$  to  $\lambda$  resp.  $\mu, \nu, \xi$ . Easy calculation shows that  $w$  can be characterized by  $w\delta = [5/2, 1/2 | \dots, 9/2, 7/2, 3/2]$  and  $w'\delta = [5/2, -1/2 | \dots, 9/2, 7/2, 3/2]$ . Because  $w'$  and  $w$  are connected by a root reflection, lemma 2.4.3 states that either  $w \leq w'$  or  $w' \leq w$  in the Bruhat ordering and there exists a sequence  $w = w_0 \rightarrow w_1 \rightarrow \dots \rightarrow w_{n-1} \rightarrow w_n = w'$ ,  $w_i \in W^p$ . Lemma 2.4.6 states  $(w_i\delta - w_{i+1}\delta)(E) \in \mathbb{N}$  for all  $i$ , where  $E$  is the grading element. But we compute  $(w\delta - w'\delta)(E) = (5/2 + 1/2) - (5/2 - 1/2) = 1$ , so the only possibility is  $n = 1$  and  $w \rightarrow w'$ . Applying 2.4.4, we see that the standard map  $M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  is nonzero.

The element  $w''$  takes  $\delta$  to  $[1/2, -5/2 | \dots, 9/2, 7/2, 3/2]$  and  $(w''\delta - w'\delta)(E) = (5/2 - 1/2) - (1/2 - 5/2) = 4$ . The length difference  $l(w'') - l(w')$  must be odd, because  $w'' = s_{\gamma}w'$  for  $\gamma = [1, 1|0, \dots, 0]$ , and a root reflection has negative determinant. So either  $w' \rightarrow w''$ , or  $w' \rightarrow w_1 \rightarrow w_2 \rightarrow w''$ . In the first case, we apply theorem 2.4.4 as before. Suppose  $w' \rightarrow w_1 \rightarrow w_2 \rightarrow w''$  and suppose, for contradiction, that the standard homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  is zero. Theorem 2.4.2 says that the true Verma modules

$$(3.1) \quad M_{\mathfrak{b}}(\nu) \subset M_{\mathfrak{b}}(s_{\alpha}\mu)$$

for some simple root  $\alpha \neq \alpha_2$ . We know that for such  $\alpha$ ,  $s_{\alpha} \in W_p$  and, because  $\mu$  is  $\mathfrak{p}$ -dominant,  $M_{\mathfrak{b}}(s_{\alpha}\mu) \subsetneq M_{\mathfrak{b}}(\mu)$ . The weight  $s_{\alpha}\mu$  is one of the following types:

- (1)  $[-1/2, 3/2 | \dots, 3, 2, 1]$  if  $\alpha = \alpha_1$
- (2)  $[3/2, -1/2 | (n-1)/2, \dots, l-1, l, \dots, 2, 1]$  if  $\alpha = \alpha_i$ ,  $2 < i < k + (n-1)/2$
- (3)  $[3/2, -1/2 | \dots, 3, 2, -1]$  if  $\alpha = \alpha_{k+(n-1)/2}$

First we show that  $\alpha \neq \alpha_1$ . If  $\alpha = \alpha_1$ , (3.1) implies that  $s_{\alpha_1}\mu - \nu = [-1, 3|0, \dots, 0]$  is a sum of positive roots, but this is not possible, as no positive root is of the form  $[-1, \text{something}]$ .

Now assume that  $s_\alpha\mu$  is of type (2). Because

$$M_{\mathfrak{b}}(w''\tilde{\lambda}) = M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(\mu) = M_{\mathfrak{b}}(w'\tilde{\lambda}),$$

$l(w') - l(w) = 3$  and  $\nu$  is not connected with  $s_\alpha\mu$  by any root reflection, it follows from Theorem 2.4.1 that there must be  $\beta_1, \beta_2$  so that

$$(3.2) \quad M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\beta_1}\nu) = M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(s_\alpha\mu).$$

Note, that the weights are  $s_\alpha\mu = [3/2, -1/2 | \dots, l-1, l, \dots, 2, 1]$  and  $\nu = s_{\beta_1}s_{\beta_2}s_\alpha\mu = [1/2, -3/2 | \dots, 2, 1]$ . In coordinates,  $s_{\beta_j}$  cannot be a (sign)-transposition interchanging an integer and a half-integer, because of the conditions  $s_\alpha\mu(H_{\beta_2}) \in \mathbb{N}$  and  $s_{\beta_2}s_\alpha\mu(H_{\beta_1}) \in \mathbb{N}$ . So, exactly one of these reflections interchanges  $(3/2, -1/2)$  to  $(1/2, -3/2)$  and the other one interchanges  $(l-1, l)$  to  $(l, l-1)$ . So either  $s_{\beta_2}s_\alpha\mu = [1/2, -3/2 | \dots, l-1, l \dots]$  or  $s_{\beta_2}s_\alpha\mu = [3/2, -1/2 | \dots, l, l-1, \dots]$ . In the first case,  $M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu) = M_{\mathfrak{b}}(s_\alpha\nu) \subsetneq M_{\mathfrak{b}}(\nu)$  ( $\nu$  is  $\mathfrak{p}$ -dominant) which contradicts (3.2). In the second case,  $M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu) = M_{\mathfrak{b}}(\mu) \subsetneq M_{\mathfrak{b}}(s_\alpha\mu)$  by (3.2), which contradicts the fact that  $M_{\mathfrak{b}}(s_\alpha\mu) \subsetneq M_{\mathfrak{b}}(\mu)$ . So  $s_\alpha\mu$  cannot be of type (2).

Similarly, we can show that  $s_\alpha(\mu)$  cannot be of type (3). But this means that (3.1) does not hold and the standard map  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  is nonzero.

Finally, note that  $w'''\delta = [-1/2, -5/2 | \dots]$ , so  $(w''\delta - w'''\delta)(E) = (1/2 - 5/2) - (-1/2 - 5/2) = 1$ , therefore  $w'' \rightarrow w'''$  and the standard homomorphism  $M_{\mathfrak{p}}(\xi) \rightarrow M_{\mathfrak{p}}(\nu)$  is nonzero.  $\square$

If  $n \neq 5$ , there are no other weights from  $P_{\mathfrak{p}}^{++} + \delta$  on the orbit of  $\tilde{\lambda}$ . In case  $n = 5$ , there are other weights  $[2, 1 | 3/2, 1/2]$ ,  $[2, -1 | 3/2, 1/2]$ ,  $[1, -2 | 3/2, 1/2]$  and  $[-1, -2 | 3/2, 1/2]$  on this orbit, but there is no nonzero homomorphism from the GVM's in the last theorem to any of these and vice versa.

**Theorem 3.1.2.** *The sequence of homomorphisms  $M_{\mathfrak{p}}(\xi) \rightarrow M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu) \rightarrow M_{\mathfrak{p}}(\lambda)$  is a complex.*

*Proof.* We want to show that the standard homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\lambda)$  is zero. This can be using theorem 2.4.2 and the facts that

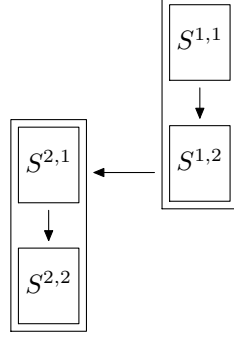
$$M_{\mathfrak{b}}([\frac{1}{2}, -\frac{3}{2} | \dots, 2, 1]) \subset M_{\mathfrak{b}}([\frac{1}{2}, \frac{3}{2} | \dots, 2, 1]) = M_{\mathfrak{b}}(s_{\alpha_1}[\frac{3}{2}, \frac{1}{2} | \dots, 2, 1]).$$

Similarly, we could show that  $M_{\mathfrak{p}}(\xi) \rightarrow M_{\mathfrak{p}}(\mu)$  is zero.  $\square$

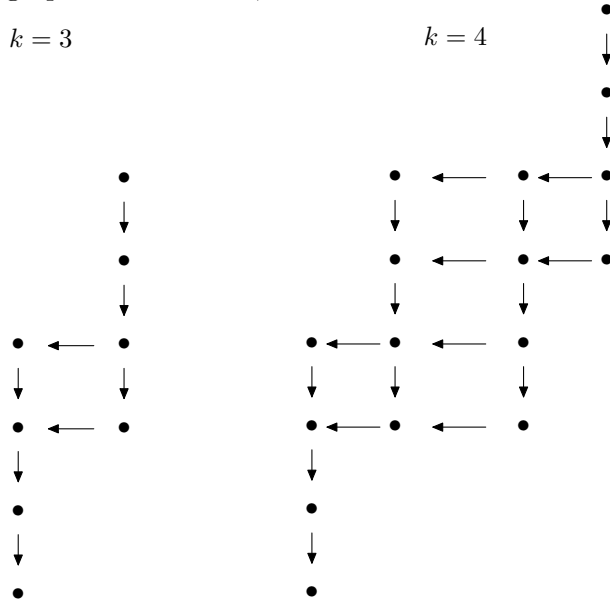
**Definition 3.1.3.** *Let us define an oriented graph  $S_k$  in the following way:  $S_1$  has 2 vertices connected by an arrow  $(\bullet \rightarrow \bullet)$ ,  $S_2$  contains 4 vertices connected linearly by arrows  $(\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet)$ . For  $k \geq 3$ ,  $S_k$  contains 2 disjoint subsets  $S^1$  and  $S^2$  of vertices so that the subgraphs  $S^1$  and  $S^2$  are both isomorphic to  $S_{k-1}$ , where  $S^1$  contains the “first” vertex and  $S^2$  the “last” one. Similarly,  $S^1$  contains 2 copies of  $S_{k-2}$ , denote them by  $S^{1,1}$  and  $S^{1,2}$  and  $S^2$  contains 2 copies of  $S_{k-2}$ , denote them by  $S^{2,1}$  and  $S^{2,2}$ . Let  $\phi$  resp.  $\psi$  be the isomorphism  $S_{k-2} \rightarrow S^{1,2}$  resp.  $S_{k-2} \rightarrow S^{2,1}$ . Then for each vertex*

$x \in S_{k-2}$  there is an arrow  $\phi(x) \rightarrow \psi(x)$  in  $S_k$ . For completeness, define  $S_0$  to be a one-point graph.

Graphically,  $S_k$  has the following structure:



We draw the graphs  $S_k$  for  $k = 3, 4$ :



**Theorem 3.1.4.** *Let  $(\mathfrak{g}, \mathfrak{p})$  and  $\lambda$  be like at the beginning of this section and let  $k \neq (n-1)/2$ . There are  $2^k$  weights from  $(P_{\mathfrak{p}}^{++} + \delta) \cap W\lambda$  and they can be assigned to the vertices of the graph  $S_k$  so that for each arrow  $\mu \rightarrow \nu$  in this graph there exists a nonzero standard homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  and each nonzero standard homomorphism between GVM's with highest weights from  $((P_{\mathfrak{p}}^{++} + \delta) \cap W\lambda) - \delta$  is a composition of these. The weight  $\lambda$  itself is assigned to the minimal vertex in  $S_k$ .*

*Proof.* The condition on a weight  $\nu = [a_1, \dots, a_k | b_1, \dots, b_{(n-1)/2}]$  to be from  $P_{\mathfrak{p}}^{++} + \delta$  is  $a_1 > \dots > a_k$ ,  $b_1 > \dots > b_{(n-1)/2} > 0$ ,  $a_i - a_j \in \mathbb{Z}$ ,  $b_i - b_j \in \mathbb{Z}$  and the  $b_i$ 's are all integers or all half-integers. Simple combinatorics implies that, if  $\nu \in P_{\mathfrak{p}}^{++} + \delta$  is on the orbit of  $\lambda$  and  $k \neq (n-1)/2$ , the only

possibility is  $\nu = [a_1, \dots, a_k | (n-1)/2, \dots, 2, 1]$ , where  $(a_1, \dots, a_k)$  is some strictly decreasing sign-permutation of  $((2k-1)/2, \dots, 3/2, 1/2)$ .

These conditions imply that there is either  $(2k-1)/2$  on the first position, or  $-(2k-1)/2$  on the  $k$ -th position and the remaining of the first  $k$  positions contains a decreasing sign-permutation of  $((2k-3)/2, \dots, 1/2)$ . This proves that there are  $2^k$  such weights. Define  $R_k$  to be the set of these weights,  $R_k^1$  to be the set of weights with  $(2k-1)/2$  on the first position and  $R_k^2$  to be the set of weights with  $-(2k-1)/2$  on the  $k$ -th position.

We will prove that the map  $i : R_{k-1} \rightarrow R_k^1$  given by  $([a_1, \dots, a_{k-1} | \dots]) \mapsto [(2k-1)/2, a_1, \dots, a_{k-1} | \dots])$  preserves the existence of nonzero standard GVM homomorphisms (i.e. there exists a nonzero standard  $M_{\mathfrak{p}_{k-1},n}(\nu) \rightarrow M_{\mathfrak{p}_{k-1},n}(\mu)$  if and only if there exists a nonzero standard  $M_{\mathfrak{p}_{k,n}}(i(\nu)) \rightarrow M_{\mathfrak{p}_{k,n}}(i(\mu))$ ), the subscripts  $k, n$  means that the rank of the Lie algebra is  $k + (n-1)/2$ .

We start with the Borel case  $\mathfrak{p} = \mathfrak{b}$ . Let  $M_{\mathfrak{b}_{k-1},n}(\nu) \rightarrow M_{\mathfrak{b}_{k-1},n}(\mu)$  be a true Verma module homomorphism. Let us denote by  $\tilde{i}$  the map  $\mathfrak{h}_{k-1,n}^* \rightarrow \mathfrak{h}_{k,n}^*$  defined by  $[a_1, \dots, a_{k-1} | b_1, \dots, b_{(n-1)/2}] \mapsto [0, a_1, \dots, a_{k-1} | b_1, \dots, b_{(n-1)/2}]$ . According to 2.4.1, there exists a nonzero homomorphism  $M_{\mathfrak{b}_{k-1},n}(\nu) \rightarrow M_{\mathfrak{b}_{k-1},n}(\mu)$  if and only if there exists a sequence  $\mu = \mu_0, \mu_1, \dots, \mu_l = \nu$  of weights connected by root reflections so that  $\mu_j - \mu_{j-1}$  is a positive integral multiple of a positive root from  $\Phi_{k-1,n}^+$  (this is the set of positive roots of  $\mathfrak{g} = \mathfrak{so}(2(k-1) + n)$ ) for all  $j$ . In this case, the sequence  $i(\mu) = i(\mu_0), i(\mu_1), \dots, i(\mu_l) = i(\nu)$  has similar properties, because  $\mu_j = s_\gamma \mu_{j-1}$  implies  $i(\mu_j) = s_{\tilde{i}(\gamma)} i(\mu_{j-1})$  and for each  $\gamma \in \Phi_{k-1,n}^+$ ,  $\tilde{i}(\gamma) \in \Phi_{k,n}^+$ . So, there exists a nonzero homomorphism  $M_{\mathfrak{b}_{k,n}}(i(\nu)) \rightarrow M_{\mathfrak{b}_{k,n}}(i(\mu))$ . On the other hand, if there exists a nonzero homomorphism  $M_{\mathfrak{b}_{k,n}}(i(\nu)) \rightarrow M_{\mathfrak{b}_{k,n}}(i(\mu))$ , it follows that there is a sequence  $i(\mu) = [(2k-1)/2, \text{something}] = i(\mu_0), i(\mu_1), \dots, i(\mu_l) = [(2k-1)/2, \text{something}]$ ,  $i(\mu_j) = s_{\gamma_j} i(\mu_{j-1})$ , so that  $i(\mu_j) - i(\mu_{j-1})$  is a positive multiple of a positive root. Therefore, the coefficient on the first position is not increasing in this sequence: so, it is constant  $(2k-1)/2$ . This means that the root reflections  $\gamma_j$  don't interchange the first coordinate with some other and the roots  $\gamma_j$  have zeros on first positions. So, there exist  $\tilde{\gamma}_j \in \Phi_{k-1,n}^+$  so that  $\tilde{i}\tilde{\gamma}_j = \gamma_j$  and we obtain that there exists a nonzero homomorphism  $M_{\mathfrak{b}_{k-1},n}(\nu) \rightarrow M_{\mathfrak{b}_{k-1},n}(\mu)$ .

It follows from Theorem 2.4.2 that the standard homomorphism  $M_{\mathfrak{p}_{k-1},n}(\nu) \rightarrow M_{\mathfrak{p}_{k-1},n}(\mu)$  is zero if and only if  $M_{\mathfrak{b}_{k-1},n}(\nu) \subset M_{\mathfrak{b}_{k-1},n}(s_{\alpha_j}\mu)$  for some simple root  $\alpha_j \neq \alpha_{k-1}$ . Then  $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\tilde{i}(\alpha_j)}i(\mu))$  follows from the previous paragraph,  $\tilde{i}(\alpha_j) \neq \alpha_k$  and the standard homomorphism  $M_{\mathfrak{p}}(i(\nu)) \rightarrow M_{\mathfrak{p}}(i(\mu))$  is zero as well. On the other hand, if  $M_{\mathfrak{p}}(i(\nu)) \rightarrow M_{\mathfrak{p}}(i(\mu))$  is zero, then  $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\alpha_i}i(\mu))$  for some simple root  $\alpha_i \neq \alpha_k$ . If  $i = 1$ , then  $M_{\mathfrak{b}_{k,n}}(i(\nu)) \subset M_{\mathfrak{b}_{k,n}}(s_{\alpha_1}i(\mu))$  implies  $s_{\alpha_1}(i(\mu)) - i(\nu)$  is a sum of positive roots. But  $i(\nu)$  contains  $(2k-1)/2$  on the first position and

$s_{\alpha_1}(i(\mu))$  contains a number strictly smaller than  $(2k-1)/2$  on the first position, what is a contradiction. Therefore,  $i > 1$  and there is a  $\alpha \in \Delta_{k-1,n}$ ,  $\alpha \neq \alpha_{k-1}$  so that  $\tilde{i}(\alpha) = \alpha_i$ . Then  $M_{\mathfrak{b}_{k-1,n}}(\nu) \subset M_{\mathfrak{b}_{k-1,n}}(s_{\alpha}\mu)$ , and the map  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  is zero as well.

We see that for any  $\mu, \nu \in R_{k-1}$ , there exists a nonzero standard GVM homomorphism  $M_{\mathfrak{p}_{k-1,n}}(\nu) \rightarrow M_{\mathfrak{p}_{k-1,n}}(\mu)$  if and only if there exists a nonzero standard homomorphism  $M_{\mathfrak{p}_{k,n}}(i(\nu)) \rightarrow M_{\mathfrak{p}_{k,n}}(i(\mu))$ . Similarly, we can define the map  $j : R_{k-1} \rightarrow R_k^2$  by  $[a_1, \dots, a_{k-1} | \dots] \mapsto [a_1, \dots, a_{k-1}, -(2k-1)/2 | \dots]$  and prove that there exists a nonzero standard GVM homomorphism  $M_{\mathfrak{p}_{k-1,n}}(\nu) \rightarrow M_{\mathfrak{p}_{k-1,n}}(\mu)$  if and only if there exists a nonzero standard homomorphism  $M_{\mathfrak{p}_{k,n}}(j(\nu)) \rightarrow M_{\mathfrak{p}_{k,n}}(j(\mu))$ .

Let us now denote the maps  $i$  and  $j$  described before by  $i_k$  and  $j_k$ , specifying the dimension of the (resulting) weights. It remains to prove that for each  $x \in R_{k-2}$  there exists a nonzero standard GVM homomorphism  $M_{\mathfrak{p}_{k,n}}(j_k i_{k-1}(x)) \rightarrow M_{\mathfrak{p}_{k,n}}(i_k j_{k-1}(x))$ . In other words, we want to show that for any decreasing sign-permutation  $(a_2, \dots, a_{k-1})$  of  $((2k-5)/2, \dots, 1/2)$ , there exists a nonzero standard homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$ , where

$$\nu = [\frac{2k-3}{2}, a_2, \dots, a_{k-1}, -\frac{2k-1}{2} | \dots, 2, 1],$$

$$\mu = [\frac{2k-1}{2}, a_2, \dots, a_{k-1}, -\frac{2k-3}{2} | \dots, 2, 1].$$

It follows from 2.4.1 that there is a homomorphism of the corresponding true Verma modules (the weights are connected by the root reflection with respect to  $[1, 0, \dots, 0, 1 | 0, \dots, 0]$ ).

There is a unique  $\mathfrak{g}$ -dominant weight  $\tilde{\lambda}$  on the orbit of  $\lambda$ :  $\tilde{\lambda} = [(n-1)/2, (n-1)/2 - 1, \dots, k, k-1/2, k-1, k-3/2, \dots, 3/2, 1, 1/2]$  in case  $(n-1)/2 \geq k$  and  $\tilde{\lambda} = [k-1/2, k-3/2, \dots, n/2, n/2-1/2, n/2-1, \dots, 1, 1/2]$  in case  $(n-1)/2 < k$ .

Let  $w$  resp.  $w'$  be the Weyl group element taking  $\tilde{\lambda}$  to  $\mu$  resp.  $\nu$ . Simple computations show that, if  $(n-1)/2 \geq k-1$ , then  $w$  takes  $\delta = \frac{1}{2}[\dots, 5, 3, 1]$  to  $\frac{1}{2}[4k-3, b_2, \dots, b_{k-1}, -(4k-7) | \dots]$  where  $(b_2, \dots, b_{k-1})$  is some decreasing sign-permutation of  $((4k-11)/2, \dots, 5/2, 1/2)$  and  $w'$  takes  $\delta$  to  $\frac{1}{2}[4k-7, b_2, \dots, b_{k-1}, -(4k-3) | \dots]$ . The difference of the grading element evaluation is then  $(w\delta - w'\delta)(E) = \frac{1}{2}((4k-3) - (4k-7) + \sum_j b_j) - \frac{1}{2}((4k-7) - (4k-3) + \sum_j b_j) = 4$ . If  $(n-1)/2 < k-1$ , then  $w$  takes  $\delta(E)$  to  $[k+n/2-1, \dots, -(k+n/2-2) | \dots]$ ,  $w'$  takes  $\delta$  to  $[k+n/2-2, \dots, -(k+n/2-1) | \dots]$  and  $(w\delta - w'\delta)(E) = 2$  in this case.

So, in either case,  $(w\delta - w'\delta)(E) \leq 4$  and, similarly as in the proof of lemma 3.1.1, either  $w \rightarrow w'$  or  $w \rightarrow w_1 \rightarrow w_2 \rightarrow w'$ . If  $w \rightarrow w'$ , we apply Theorem 2.4.4 and see that there is a nonzero standard homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$ .

Let  $w \rightarrow w_1 \rightarrow w_2 \rightarrow w'$  and assume, for the sake of contradiction, that the standard map  $M_{\mathfrak{p}}(w'\tilde{\lambda}) \rightarrow M_{\mathfrak{p}}(w\tilde{\lambda})$  is zero. Therefore,

$$(3.3) \quad M_{\mathfrak{b}}(\nu) = M_{\mathfrak{b}}(w'\tilde{\lambda}) \subset M_{\mathfrak{b}}(s_{\alpha}w\tilde{\lambda}) = M_{\mathfrak{b}}(s_{\alpha}\mu)$$

for some simple root  $\alpha \neq \alpha_k$ .

The weight  $s_{\alpha}(\mu)$  is one of the following types:

- (1)  $[a_2, (2k-1)/2, \dots, a_{k-1}, -(2k-3)/2 | \dots, 3, 2, 1]$  if  $\alpha = \alpha_1$
- (2)  $[(2k-1)/2, \dots, a_l, a_{l-1}, \dots, -(2k-3)/2 | \dots]$  if  $\alpha = \alpha_j$ ,  $1 < j < k-1$
- (3)  $[(2k-1)/2, \dots, -(2k-3)/2, a_{k-1} | \dots]$  if  $\alpha = \alpha_{k-1}$
- (4)  $[(2k-1)/2, \dots, -(2k-3)/2 | (n-1)/2, \dots, l-1, l, \dots, 2, 1]$  if  $\alpha = \alpha_j$ ,  $k < j < k + (n-1)/2$
- (5)  $[(2k-1)/2, \dots, -(2k-3)/2 | \dots, 3, 2, -1]$  if  $\alpha = \alpha_{k+(n-1)/2}$

First we show that it is not of type (1). If  $\alpha = \alpha_1$ , (3.3) implies  $s_{\alpha_1}(\mu) - \nu$  is a sum of positive roots, i.e.

$$[a_2, (2k-1)/2, \dots, -(2k-3)/2 | \dots] - [(2k-3)/2, a_2, \dots, -(2k-1)/2 | \dots] \in \mathbb{N}\Phi^+,$$

where  $a_2 \leq (2k-5)/2$ . But the difference cannot be obtained as a sum of positive roots, because it contains a negative number  $a_2 - (2k-3)/2$  on the first position.

Now assume that  $s_{\alpha}(\mu)$  is of type (2) – (5). Because

$$M_{\mathfrak{b}}(w'\tilde{\lambda}) = M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\alpha}\mu) \subsetneq M_{\mathfrak{b}}(\mu) = M_{\mathfrak{b}}(w\tilde{\lambda}),$$

$l(w') - l(w) = 3$  and  $\nu$  is not connected to  $s_{\alpha}(\mu)$  by any root reflection, it follows from Theorem 2.4.1 that there must be  $\beta_1, \beta_2$  so that

$$(3.4) \quad M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\beta_1}\nu) = M_{\mathfrak{b}}(s_{\beta_2}s_{\alpha}\mu) \subsetneq M_{\mathfrak{b}}(s_{\alpha}\mu)$$

Similarly as in the proof of lemma 3.1.1, we will show that  $\alpha$  cannot be of type (2) – (5). Let  $\alpha$  be of type (2), i.e.

$$\begin{aligned} s_{\alpha}(\mu) &= [(2k-1)/2, \dots, a_l, a_{l-1}, \dots, -(2k-3)/2 | \dots], \\ \nu &= [(2k-3)/2, \dots, a_{l-1}, a_l, \dots, -(2k-1)/2 | \dots]. \end{aligned}$$

The root reflections  $s_{\beta_1}$  and  $s_{\beta_2}$  cannot interchange an integer with a half-integer, because of the integrality conditions  $s_{\alpha}(\mu)(H_{\beta_2}) \in \mathbb{N}$  and  $s_{\beta_2}s_{\alpha}(\mu)(H_{\beta_1}) \in \mathbb{N}$ . There are two possibilities: either  $s_{\beta_2}$  interchanges  $a_l$  with  $a_{l-1}$  and  $s_{\beta_1}$  sign-interchanges  $((2k-1)/2, -(2k-3)/2)$  with  $((2k-3)/2, -(2k-1)/2)$  on the particular positions, or  $s_{\beta_2}$  sign-interchanges  $((2k-1)/2, -(2k-3)/2)$  with  $((2k-3)/2, -(2k-1)/2)$  and  $s_{\beta_1}$  interchanges  $a_l$  with  $a_{l-1}$ . In the first case,  $\beta_2 = \alpha$  and (3.4) implies  $M_{\mathfrak{b}}(\mu) \subsetneq M_{\mathfrak{b}}(s_{\alpha}\mu)$ , which contradicts the fact that  $M(s_{\alpha}\mu) \subsetneq M(\mu)$  for a simple root  $\alpha \neq \alpha_k$  and  $\mu \in P_{\mathfrak{p}}^{++} + \delta$ . In the second case,  $\beta_1 = \alpha$  and (3.4) implies  $M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\alpha}\nu)$ , which also contradicts  $M_{\mathfrak{b}}(s_{\alpha}\nu) \subsetneq M_{\mathfrak{b}}(\nu)$ .



Let  $\alpha$  be of type (3), i.e.

$$\begin{aligned} s_\alpha(\mu) &= [(2k-1)/2, \dots, -(2k-3)/2, a_{k-1} | \dots], \\ \nu &= [(2k-3)/2, \dots, a_{k-1}, -(2k-1)/2 | \dots] \end{aligned}$$

If either  $\beta_1 = \alpha$  or  $\beta_2 = \alpha$ , we get contradiction similarly as in case (2). But there is no other possibility, because the  $a_{k-1}$  on the  $k$ -th position has to move somehow to the  $(k-1)$ -th position: if  $\beta_2$  would fix it, then  $\beta_1 = \alpha$ , if  $\beta_2$  would take it to the  $(k-1)$ -th position, then  $\beta_2 = \alpha$  and if  $\beta_2$  would take it (possibly with a minus sign) to the  $l$ -th position for  $l \neq k, k-1, 1$ , then  $\beta_1$  has to (sign-) interchange the  $l$ -th and  $(k-1)$ -th position, so  $s_{\beta_1}s_{\beta_2}$  would fix the  $(2k-1)/2$  on the first position, which is impossible. The last possibility is  $l = 1$ : this would mean that  $\beta_2$  takes  $a_{k-1}$  to the first position (possibly with a minus sign), but  $|a_{k-1}| < (2k-3)/2$  implies that  $s_{\beta_2}s_\alpha(\mu)$  has a smaller number on the first position as  $\nu$  and  $s_{\beta_2}s_\alpha(\mu) - \nu$  is not expressible as a sum of positive roots. This contradicts  $M_{\mathfrak{b}}(\nu) \subsetneq M_{\mathfrak{b}}(s_{\beta_2}s_\alpha\mu)$ .

In case (4), we have

$$\begin{aligned} s_\alpha(\mu) &= [(2k-1)/2, \dots, -(2k-3)/2 | (n-1)/2, \dots, l-1, l, \dots, 2, 1] \\ \nu &= [(2k-3)/2, \dots, -(2k-1)/2 | (n-1)/2, \dots, l, l-1, \dots, 2, 1] \end{aligned}$$

Because the reflections with respect to  $\beta_1, \beta_2$  cannot interchange an integer and a half-integer, it follows that one of them interchanges  $l$  with  $l-1$ , so either  $\beta_1 = \alpha$  or  $\beta_2 = \alpha$  and we get a contradiction as in case (2). The same happens in case (5).

In either case, we get a contradiction, so the standard map  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  is nonzero.

So, we can assign weights from  $R_k$  to the vertices of the graph  $S_k$  so that we assign the weights from  $R^1$  to  $S^1$ , the weights from  $R^2$  to  $S^2$  and the proof follows by induction.

Finally, it is easy to check that any possible nonzero standard GVM homomorphisms on the orbit is a composition of the homomorphisms described above by reducing this problem to true Verma module homomorphisms and considering theorem 2.4.1.  $\square$

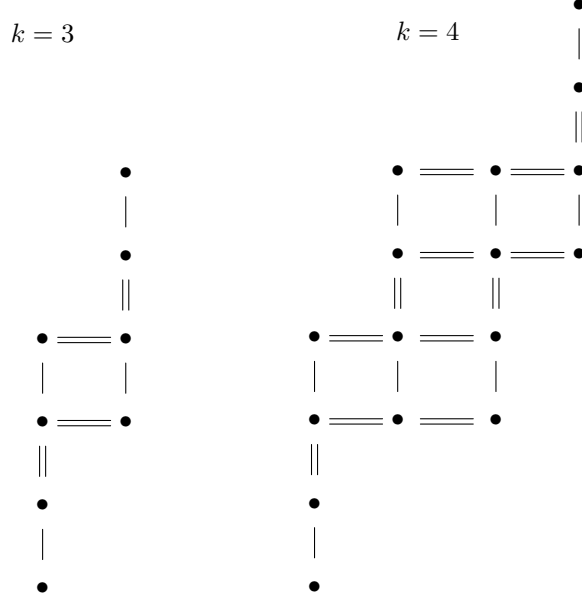
In case  $k = (n-1)/2$ , all the GVM homomorphisms described in the last theorem exist as well, but the whole orbit contains also weights of type  $[\dots, 2, 1 | (2k-1)/2, \dots, 3/2, 1/2]$ . There is no nonzero GVM homomorphism  $M_{\mathfrak{p}}(\nu) \rightarrow M_{\mathfrak{p}}(\mu)$  where  $\mu$  is of such type and  $\nu$  of the type  $[\dots, 3/2, 1/2 | \dots, 2, 1]$  (or opposite).

### 3.2. Orders of the operators.

**Theorem 3.2.1.** *All the operators dual to the homomorphisms described in theorem 3.1.4 have order 1 or 2. For any  $k$ , the connecting operators*

$\phi(x) \rightarrow \psi(x)$  (described in definition 3.1.3) have order 2 and the graph homomorphisms  $S_{k-1} \rightarrow S_k^1$  and  $S_{k-1} \rightarrow S_k^2$  respect orders. This determines, by induction, all the order of all the operators.

If we draw a line for first order operators and a double-line for second order operators in the diagrams, we obtain the following pictures:



*Proof.* Recall that the action of a weight on the grading element is

$$[a_1, \dots, a_k | b_1, \dots, b_{(n-1)/2}](E) = \sum_j a_j.$$

Applying theorem 2.5.1 and the knowledge of the highest weights of the particular representations, we see that

$$\begin{aligned} & [(\frac{2k-1}{2}, a_2, \dots, a_{k-1}, -\frac{2k-3}{2} | \dots)](E) - [\frac{2k-3}{2}, a_2, \dots, a_{k-1}, -\frac{2k-1}{2} | \dots](E) = \\ & = (\frac{2k-1}{2} - \frac{2k-3}{2}) - (\frac{2k-3}{2} - \frac{2k-1}{2}) = 2, \end{aligned}$$

so the “connecting” operators are of second order. The other operators are of first order, because

$$\begin{aligned} & [a_1, \dots, a_{j-1}, \frac{1}{2}, a_{j+1}, \dots | \dots](E) - [a_1, \dots, a_{j-1}, -\frac{1}{2}, a_{j+1} \dots | \dots](E) = \\ & \frac{1}{2} - (-\frac{1}{2}) = 1. \end{aligned}$$

□

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